数理解析研究所講究録
209巻 1974年 125-134

On certain L^2-well posed mixed problems for hyperbolic system of first order

by

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1. Introduction and Theorem.

Let P be a x_0-strictly hyperbolic $2p \times 2p$-system of differential operators of first order defined over a C^∞-cylinder $R^1 \times \Omega \subset R^{n+1}$. Let B be a $p \times 2p$-system of functions defined on the boundary Γ of $R^1 \times \Omega$. We consider the following mixed problems under certain conditions:

$$P(x, D)u = f \quad x \in R^1 \times \Omega \quad (x_0 > 0),$$
$$B(x)\, u = g \quad x \in \Gamma \quad (x_0 > 0),$$
$$u = h \quad \text{on } x_0 = 0$$

where $\sqrt{-1}\, D = (\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \dots\dots, \frac{\partial}{\partial x_n})$.

For the sake of simplicity of descriptions, we may only consider the case where $\Omega = \{ x_n > 0\}$, by the localization process. Then our assumptions are the following:

(I). α) The coefficients of P and B are real, belong to $C^\infty (R^1 \times \overline{\Omega})$ and constant outside some compact set of $R^1 \times \overline{\Omega}$.

β) For P, it satisfies the # condition with respect to Γ and for fixed real (x, τ, σ) there is at most one real double

root λ of $|P|\ (x, \tau, \sigma, \lambda) = 0$ where $x \in \Gamma$.
Furthermore it is non-characteristic with respect to Γ and it is normal, i.e.

$$|P|\ (x, 0, \sigma, \lambda) \neq 0$$

for any real $(\sigma, \lambda) \neq 0$.

γ) The p row-vectors of $B(x)$ are linearly independent, where $x \in \Gamma$.

(II). α) If the Lopatinsky determinant $R(x_0, \tau_0, \sigma_0) = 0$ for a real point (x_0, τ_0, σ_0) such that there is no real double roots λ of $|P|(x_0, \tau_0, \sigma_0, \lambda) = 0$, then

$$|R\ (x_0, \tau_0 - i\gamma, \sigma_0)| \geq 0(\gamma^1) \quad (\gamma > 0).$$

Furthermore if there is at least one real simple root $\lambda\ (x_0, \tau_0, \sigma_0)$, the zero set of $R(x, \tau \pm i\gamma, \sigma)$ in some neighbourhood $U(x_0, \tau_0, \sigma_0)$ is in the set $\{\gamma = 0\}$.

β) If $R(x_0, \tau_0, \sigma_0) = 0$ for a real point (x_0, τ_0, σ_0) such that there are real double roots λ of $|P|\ (x_0, \tau_0, \sigma_0, \lambda) = 0$, then

$$|R(x_0, \tau_0 - i\gamma, \sigma_0)| \geq 0(\gamma^{\frac{1}{2}}) \quad (\gamma > 0).$$

Furthermore if there is at least one real simple root λ, the rank of the Hessian of $R(x, \tau, \sigma)$ at its zeros in some $U(x_0, \tau_0, \sigma_0)$ is equal to

$$\text{codim. of } \{R(x, \tau, \sigma) = 0\} \text{ in } R^{2n}.$$

Where the zero set of $R(x, \tau, \sigma)$ in some $U(x_0, \tau_0, \sigma_0)$ is preassumed to be a regular submanifold of R^{2n}.

γ) Moreover, if there is at least one non-real root λ of $|P|(x_0, \tau_0, \sigma_0, \lambda) = 0$ for the point (x_0, τ_0, σ_0) which satisfies the condition β), then for some smooth and non-singular matrix $S(x, \tau-i\gamma, \sigma)$ with $\gamma \geqq 0$ defined on some $U(x_0, \tau_0, \sigma_0)$ the corresponding reflection coefficient $b_{II\,II}(x, \tau, \sigma)$ is real whenever τ is real and $R(x, \tau, \sigma) \neq 0$ (For definitions, see §2).

(III). Any constant coefficients problems frozen the coefficient at boundary are L^2-well posed.

Then we have the following

Theorem. Under assumptions (I), (II), (III), the mixed problem is L^2-well posed.

The aim of the present note is to describe the outline of our proof of the above assertion. Here we use essentially the conception of reflection coefficients ([1], [2]) and modifying Kreiss' consideration [4] we make use of the localization of the characterization for L^2-well posed mixed problem of order two. ([1], [3] and [7])

2. The outline of the proof.

Considering the assumption (I) let $S(x, \tau-i\gamma, \sigma)$ $(\gamma \geqq 0)$ be a smooth, non-singular matrix defined on some neighbourhood $U(x_0, \tau_0, \sigma_0)$ such that

$$S^{-1}PS = ED_n - A(x, \tau - i\gamma, \sigma)$$

where

$$A = \begin{pmatrix} \lambda_I^+ & & & & \\ & \lambda_I^- & & & \\ & & A_{II} & & \\ & & & A_{III}^+ & \\ & & & & A_{III}^- \end{pmatrix},$$

$$\lambda_I^{\pm} = \begin{pmatrix} \ddots & & \\ & \lambda_i^{\pm} & \\ & & \ddots \end{pmatrix}, \ i \in I, \ |I| = r,$$

$\lambda_i^{\pm}$ are real, and $\mathrm{Im}\ \lambda_i^+$ ($\mathrm{Im}\ \lambda_i^-$) > 0 (< 0) respectively if $\gamma > 0$.
Next for $\tau_0 = \tau_0(x, \sigma)$

$$A_{II}(x, \tau_0, \sigma) = \begin{pmatrix} a(x, 0, \sigma) & 1 \\ 0 & a(x, 0, \sigma) \end{pmatrix}.$$

Here we may restrict ourself to the case where the eigenvalue of $A_{II}(x, \tau, \sigma)$ are described by the following form in some $U(x_0, \tau_0, \sigma_0)$

$$\lambda_{II}^{\pm} = a(x, \zeta, \sigma) \mp \sqrt{\zeta}\, b(x, \zeta, \sigma) \qquad (\sqrt{1} = 1),$$

$a(x, \zeta, \sigma)$, $b(x, \zeta, \sigma)$ are real when ζ is real, $b(x, \zeta, \sigma) \neq 0$, $\tau_0 = \tau_0(x_0, \sigma_0)$, $\tau = \zeta + \tau_0(x, \sigma)$ and $\tau_0(x, \sigma)$ is real and positive.

Furthermore $A_{III}^{\pm}$ have only non-real eigenvalues for any $\gamma \geq 0$
and the ones of A_{III}^+ have positive imaginary parts.

Let $BS' = (V_I^+, V_I^-, V_{II}', V_{II}'', V_{III}^+, V_{III}^-)$.

Where $V_{I}^{\pm}$ are $(p\times r)$-matrices, $V_{II}^{\prime}$, $V_{II}^{\prime\prime}$ are p-vectors and $V_{III}^{\pm}$ are $(p\times s)$-matrices respectively $(2r+2+2s = 2p)$.

Let $S_{II} = \begin{pmatrix} 1 & 0 \\ \dfrac{\lambda_{II}^{+}-h_{11}\zeta-a}{1+h_{12}\zeta}, & 1 \end{pmatrix}$, $a = a(x, 0, \sigma)$; *

and let

$$S' = \begin{pmatrix} E_{2r}, & & \\ & S_{II}, & \\ & & E_{2s}, \end{pmatrix},$$

where h_{ij} are the functions derived from $A_{II}(x, \ -i\gamma, \sigma)$. Furthermore we denote $B\cdot S\cdot S'$ by

$$(V_{I}^{+}, V_{I}^{-}, V_{II}^{+}, V_{II}^{-}, V_{III}^{+}, V_{III}^{-}) (x, \tau, \sigma).$$

Then from our assumptions we obtain the following Lemmas. In particular from (I). γ), (II). α) and (III), we see the following

Lemma 2. 1 If for real (x_0, τ_0, σ_0) there exist no real double roots λ, then there is a neighbourhood $U(x_0, \tau_0, \sigma_0)$ where

i) For some $V_{3,i}^{-}$ the determinant

$$|V_{I}^{+}, V_{31}^{+}, \cdots, V_{3,i-1}^{+}, V_{3,i}^{-}, V_{3,i+1}^{+}, \cdots, V_{3,s}^{+}| \neq 0$$

where $V_{III}^{+} = (V_{3,1}^{+}, \cdots, V_{3.s})$, $s = p - \gamma$, $V_{3,i}^{+}$ are p-column vectors

(Here after let $i = 1$).

ii) For some $V_{3,1}^{+}$ it belongs to the linear subspace $L(V_{3,2}^{+},\cdots, V_{3,s}^{+})$ spanned by the vectors $V_{3,2}^{+},\cdots, V_{3,s}^{+}$.

iii) The column vectors of V_{I}^{-} belong to $L(V_{I}^{+}, V_{3,2}^{+},\cdots, V_{3,s}^{+})$.

But ii) and iii) are only valid at the points $\in U(x_0, \tau_0, \sigma_0)$ such that the Lopatinsky det. $|V_{I}^{+}, V_{II}^{+}|\ (x, \tau, \sigma) = c(\tau-\dot{\tau}(x, \sigma)) = 0$ $(c \neq 0)$ and where $\tau(x, \sigma)$ is real whenever V_{I}^{+} present.

From (II). β) we see the following

Lemma 2. 2 Let (x_0, τ_0, σ_0) be a real point such that there exists a real double root λ. Let $|V_{I}^{+}, V_{II}^{+}, V_{III}^{+}|\ (x_0, \zeta, \sigma_0) = 0$, where we consider ζ as a new variable instead of τ. Then

i) $\zeta = 0$.

ii) Let $\zeta^{\frac{1}{2}} = \eta$, then

$$|V_{I}^{+}, V_{II}^{+}, V_{III}^{+}| = C(\eta-\eta(x, \sigma)) \qquad (c \neq 0)$$

in some $U(x_0, \tau_0, \sigma_0)$, where $\eta(x, \sigma)$ may take ~~complex~~ values. $\not\wedge$

Under the assumption of Lemma 2. 2 we see the following Lemmas.

Lemma 2. 3 i) The reflection coefficient

$$b_{II\,I}(x_0, -i\gamma, \sigma_0) = \frac{|V_{I}^{+}, V_{II}^{-}, V_{III}^{+}|}{|V_{I}^{+}, V_{II}^{+}, V_{III}^{+}|}\ (x_0, -i\gamma, \sigma_0)$$

$$= O(\gamma^{-\frac{1}{2}}) \qquad (\gamma > 0).$$

ii) Let $Q(x, \zeta, \sigma)$ be $\dfrac{a_{11} + a_{12}b_{\mathrm{II}}}{a_{12} + a_{22}b_{\mathrm{II}}}$, then

it is $\dfrac{|V_{\mathrm{I}}^{+}, V_{\mathrm{II}}', V_{\mathrm{III}}^{+}|}{|V_{\mathrm{I}}^{+}, V_{\mathrm{II}}'', V_{\mathrm{III}}^{+}|}$, where $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = S_{\mathrm{II}}^{-1}$.

Now from Lemma 2. 3 and (III) we obtain the following

Lemma 2. 4

i) $|V_{\mathrm{I}}^{+}, V_{\mathrm{II}}'', V_{\mathrm{III}}^{+}| \neq 0$.

ii) $V_{\mathrm{II}}' \in L(V_{\mathrm{III}}^{+})$ on $\zeta = \eta(x, \sigma) = 0$.

iii) $V_{\mathrm{I}}^{-} \in L(V_{\mathrm{I}}^{+}, V_{\mathrm{III}}^{+})$ on $\zeta = \eta(x, \sigma) = 0$.

iv) $V_{\mathrm{II}}' - QV_{\mathrm{II}}'' \in L(V_{\mathrm{I}}^{+}, V_{\mathrm{III}}^{+})$.

From (II). β), γ), (III) and the definition of Q we see the following

Lemma 2. 5 i) The above defined $Q(x, \zeta, \sigma)$ take only real values, when ζ is real.

ii) $\zeta = 0$, $Q(x, 0, \sigma) = 0$ are equivalent to $R(x, \zeta, \sigma) = 0$ for $\mathrm{Im}\,\zeta \leq 0$.

iii) $-Q(x, 0, \sigma) \geq 0$.

From Lemma 2. 4 we obtain the following

Lemma 2. 6 For (x, ζ, σ) belonging to some $U(x_0, \tau_0, \sigma_0)$,

$$g = (V_I^+, V_{II}'', V_{III}^+)\begin{pmatrix} U_I^+ + (\zeta K_{III}' + K_{III}'')U' + K_{II}U_I^- \\ U_{II}'' + QU_{II}' + (\zeta K_{III}' + K_{III}'')U_I^- \\ U_{III}^+ + K_{III\,I}U_I^- + K_{III\,II}U_{II}' \end{pmatrix}$$

$$+ V_{III}^- \begin{pmatrix} 0 \\ 0 \\ U_{III}^- \end{pmatrix},$$

where $u = (U_I^+, U_I^-, U_{II}', U_{II}'', U_{III}^+, U_{III}^-)$.
Moreover the components of K_{III}'' and K_{III}'' are zero, whenever $\zeta = 0$ and $\eta(x, \sigma) = 0$.

From Lemma 2. 1 we obtain an a priori L^2-estimate in the case where there is no double root λ. On the other hand if there is at least one double root λ, we see from Lemma 2. 5 and by some modifications of Kreiss' method that the problem $((D_n - A_{II})u = f,\ u'' + Qu' = g)$ has an a priori estimate

$$\|(D_n - A_{II})u\|_{0,\gamma} + \langle\!\langle g \rangle\!\rangle_{\frac{1}{2},\gamma} \geq C\gamma\|u\|_{0,\gamma} \qquad (C > 0)$$

where supp $u \subset U(x_0)$, spectrum of u with respect to $x_0, \cdots, x_{n-1}$ $\subset U(\tau_0, \sigma_0)$. Then from the method of the proof of the above estimate and from Lemma 2. 6, we obtain a similar estimate in this case. Here we use the fact that the components k of K_{III}', K_{III}' has the following form: in some $U(x_0, \tau_0, \sigma_0)$

$$k(x, \zeta, \sigma) = \tilde{k}(x, 0, \sigma) + \zeta\tilde{\tilde{k}}(x, 0, \sigma) + O(|\zeta|^2),$$

$$|\tilde{k}(x, 0, \sigma)|^2 \leq K|Q(x, 0, \sigma)| \qquad (K > 0)$$

which follows from the last assumption of (II), (β). Furthermore our assumptions are valid for the dual problem and hence a priori estimate for that problem is also obtained. Thus our proof is complete ([6]).

Remark (1) The conditions (I), (II), (III) are invariant for certain coordinate transformation. Hence Theorem is applicable for problems defined on any smooth $R^1 \times \Omega$.

(2) The condition (II), γ) should be omitted, but we have many examples which satisfy the condition.

References

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